Pergamon

Int. J. Solids Structures Vol. 32, No. 2, pp. 165–172, 1995

0020–7683/95 $9.50 + .00

0020–7683(94)00142–1

NEW CLASS OF CREEP–RELAXATION FUNCTIONS

S. HAZANOV
Department of Materials, Swiss Federal Institute of Technology, EPFL, Ecublens, MX-G, 1015 Lausanne, Switzerland

(*Received* 24 *November* 1993; *in revised form* 10 *June* 1994)

Abstract—The thermodynamic analysis of viscoelastic constitutive laws makes it possible to estimate the class of admissible creep–relaxation functions. This set of functions turns out to be larger than it was traditionally supposed. Except for the well known positive definite, monotone decreasing functions, the new class also includes the so-called non-traditional functions, such as locally non-monotone and even non-positive ones, hence involving the effect of the so-called negative viscosity.

It is shown that generalized creep–relaxation functions turn out to be effective in the phenomenologic modelling of the complicated materials behaviour, like that in homogenization of heterogeneous bodies and in modelling of the wave propagation and of the negative Poisson ratio phenomena in composites.

New directions for fundamental and applied research, that are considered with the thermodynamic admissibility of viscoelastic models, are discussed. Examples of the application of generalized creep–relaxation functions are given, in particular for the dynamics of composites and for the thermomechanical modelling of wood.

1. INTRODUCTION

Though being thoroughly studied throughout the last century, the classic theory of linear viscoelasticity still contains some ambiguous points, some fundamental problems to be resolved. One of them is the estimation of the limits of the class of admissible creep–relaxation functions in the constitutive viscoelastic laws, usually taken in the convolution form.

Traditionally, only positive definite monotone decreasing functions are used in the applications; these are exponential or power functions (with a weak integrable singularity at the zero point). The historical reason for this is that linear viscoelasticity was initiated with the study of polymers' behaviour, where exponential kernels are usually quite sufficient. Later, when dealing with the creep of metals, the power (weak singular) kernels of heredity were put into use. Thus, the only one reasonable thermodynamic argument for the mentioned tradition is the fading memory condition, which is automatically fulfilled for positive monotone decreasing functions.

The situation has changed recently, with the creation and development of new complicated materials and respectively of new branches of solid mechanics, such as mechanics of composites. Classic viscoelastic models turned out to be insufficient for these new materials and the necessity for new models has arisen. Furthermore, this made a thorough thermodynamic analysis of the problem of admissible creep–relaxation functions necessary.

2. MECHANICAL ADMISSIBILITY

As it was already formulated above, the problem under consideration is to analyse the class of thermodynamically admissible creep and relaxation functions (tensors f and r of the fourth order) in the classic viscoelastic constitutive equations, relating the deformation and stress tensors ε and σ:

$$\varepsilon = f * \sigma \quad \sigma = r * \varepsilon \tag{1}$$

where operator * is the Stieltjès convolution.

The first and the only one reasonable energy restriction in such a formulation is the positivity of the work W during the deformation cycle:

$$W = \int_0^t \sigma \, \mathrm{d}\varepsilon > 0. \tag{2}$$

As it was shown in Breuer and Onat (1962), a sufficient condition to guarantee the positivity of the work is to assure the positivity of the cos–Fourier transform of the function $(r(t) - r(\infty))$, where $r(\infty)$ is the long-term modulus:

$$(r - r(\infty))_c = \sqrt{\frac{2}{\pi}} \int_0^\infty (r(\tau) - r(\infty)) \cos \omega\tau \, \mathrm{d}\tau > 0. \tag{3}$$

By appealing to the classic Bochner theorem from the probability theory, Breuer and Onat (1962) state that, in order to satisfy the criterion inequality (3), it is sufficient to choose the relaxation function $r(t)$ from the class of non-negative, monotone decreasing, convex functions. So there is little wonder that, in practical and theoretical computations, engineers were intuitively doing the data fitting in terms of such positive monotone decreasing functions, the most typical of which is the exponential one.

However, in reality, as it can be easily verified, the class of functions defined by inequality (3) is wider and also includes numerous non-monotone and even locally negative relaxation functions, for instance such as

$$(A \cos(at) + B \sin(bt)) \exp(-ct), \quad A \exp(-at) - B \exp(-bt), \quad J_0(at), \ldots. \tag{4}$$

This has already been noted as a mathematical curiosity in Gurtin and Herrera (1965) and later in Hazanov (1976) where its thermodynamic admissibility and possible applications were analysed.

From the physical point of view, it is important that all such generalized relaxation functions satisfy the condition of positivity of the global viscosity of the body:

$$\eta = \int_0^\infty r(t) \, \mathrm{d}t > 0. \tag{5}$$

This follows directly from inequality (3) when variable ω tends to 0. For creep functions a similar admissibility condition can be easily established.

Expressions (4) also show that the generalized relaxation functions introduced above satisfy the fading memory condition. The latter is understood here in a generalized, integral sense, but not in a local sense.

3. DISSIPATION RESTRICTIONS

Work positivity is the only one universal energy condition which is valid for all the viscoelastic models. All the other thermodynamic demands involve the dissipation rate and hence depend on the choice of a form of the free energy, which in turn demands introduction of some additional hypothesis.

An exception is the case of harmonic loading, when the dissipation power of the body is directly related to the imaginary component of the complex moduli, which is proportional to the cos–Fourier transform of the relaxation function [see, for example, Christensen (1971)]. Hence, all the thermodynamic conditions for such a quasi-stationary loading are contained in inequality (3).

Let us turn to an arbitrary loading. One of the forms of the free energy most frequently used in practice is the Staverman–Schwarzl phenomenological expression, established originally in 1953 for exponential relaxation functions and given, for instance, in Christensen (1971):

$$F = 1/2 \int_0^t \int_0^t r(2t-u-v)\,\mathrm{d}\varepsilon(v)\,\mathrm{d}\varepsilon(u). \tag{6}$$

In this case, the Clausius–Duhem inequality gives the following expression for the dissipation power D, as is shown, for instance, in Rabotnov (1977):

$$D = -\int_0^t \int_0^t r'(2t-u-v)\,\mathrm{d}\varepsilon(v)\,\mathrm{d}\varepsilon(u) > 0. \tag{7}$$

Let us recall the spectral representation of the relaxation function (tensor):

$$r(t) = \int_0^\infty \rho(\mu)\exp(-\mu t)\,\mathrm{d}\mu + r(\infty). \tag{8}$$

Then, by substituting eqn (8) into eqn (7), one obtains the expression

$$D = \int_0^\infty \mu\rho(\mu)\left(\int_0^t \mathrm{e}^{-\mu(t-u)}\,\mathrm{d}\varepsilon(u)\right)^2 \mathrm{d}\mu > 0. \tag{9}$$

Inequality (9) leads to the obvious conclusion that the positivity of the viscoelatic dissipation power is quite assured by the positivity of the relaxation function spectrum $\rho(\mu)$:

$$\rho(\mu) > 0. \tag{10}$$

Thus, we have obtained another condition (a dissipative one) for the thermodynamic admissibility of creep–relaxation functions.

Evidently, eqn (6) is not unique; there are other alternative free energy expressions, like that of the quadratic functional presented in Rabotnov (1977). However, we will no longer stay on the concept of the dissipation positivity, because in the actual state-of-the-art of the viscoelasticity theory, all the hypotheses concerning the form of the free energy functional are rather subjective. Moreover, the computation of the relaxation function spectrum poses a lot of technical problems. Thus, in our view, the positivity of the work discussed above in inequalities (2) and (3) is, for the moment, the only one effective criterion for the verification of the thermodynamic admissibility of creep–relaxation functions.

4. MECHANICAL MODELS

The non-standard creep–relaxation functions and kernels of heredity introduced above can be obtained even in terms of the classic mechanical spring–dashpot models, for instance by introducing into them inertial terms. Let us first take a Maxwell element and introduce into it a mass M, placing it between the spring and the dashpot. Then, by writing down the equation of motion for this mass, we will arrive at the constitutive law of the constructed system. For large values of the dashpot viscosity, this law gives the relaxation kernel of the system in the form

$$G(t) = A(\exp(-at) - \exp(-bt)), \tag{11}$$

which indicates a non-monotone function. Correspondingly, for small values of the dashpot viscosity, the creep kernel in the constitutive equation of the system will be

$$K(t) = A\exp(-at)\sin bt, \tag{12}$$

which indicates an oscillating, non-monotone and locally even negative function.

This list can be continued, but we will no longer remain at this point as we prefer the Volterra operator concept as the basic for the construction of viscoelastic theories. The example presented here was a mere illustration of the relation between the introduced new class of creep–relaxation functions with the classical mechanical modelling approach.

5. DYNAMICS OF COMPOSITES

One of the main modern tendencies in modelling the mechanical behaviour of composites is its homogenization with the subsequent evaluation of the effective material properties. In this manner, wave propagation in laminated composites is often described by homogeneous viscoelastic Maxwell models, as is done in Barker (1971) whereas the kinetics of wood is often described by viscoelastic models with parabolic elements, as is proposed in Huet (1988). However, what concerns the dynamics of composites, in many cases classical homogeneous viscoelastic models, cannot provide a good quantitative or even qualitative description of experimental phenomena. Let us study this point in detail.

5.1. *Geometric dispersion*

Such an insufficiency of the classic viscoelastic homogenization has a place, for instance, with the well known effect of geometric dispersion of harmonic waves in laminated composites. The experimental data in this case clearly show the existence of two branches and of a resonance point on the curve of the phase velocity (see Fig. 1), though this fact contradicts the corresponding diagrams for homogeneous bodies.

These experimental results are presented in Sutherland and Lingle (1972) and were received by sending harmonic signals through plates ($7.6 \times 7.6 \times 1.01$ cm) of Al-W fibre composite. The main result of the data processing was the graph of the relation between the phase velocity C_p and the frequency ω. Figure 1 indicates a strong intensification of the geometric dispersion near the resonance point (3.5 MHz)

Simple analysis shows that classic viscoelastic models cannot, in principle, describe this effect, even qualitatively. Indeed, let us analyse the harmonic vibrations (U—displacement)

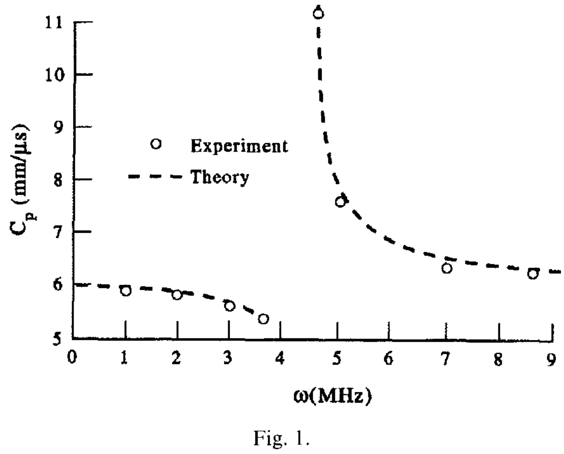

Fig. 1.

$$U(x,t) = U_0 \exp(k_2 x) \exp(i(wt - k_1 x)) \tag{13}$$

of a viscoelastic Volterra body:

$$E\varepsilon = \sigma + \int_{-\infty}^{t} K(t-u)\sigma(u)\,du. \tag{14}$$

Equations (13) and (14) lead to the following relation for the phase velocity C_p:

$$C_p = \frac{\omega}{k_1} = \frac{c\sqrt{2}}{\sqrt{E(J_1 + \sqrt{(J_1)^2 + (J_2)^2})}}, \tag{15}$$

where c is the elastic wave velocity and J_1 and J_2 are, respectively,

$$J_1 = \left(1 + \int_0^\infty K(t)\cos\omega t\,dt\right)\bigg/E, \quad J_2 = \left(\int_0^\infty K(t)\sin\omega t\,dt\right)\bigg/E. \tag{16}$$

Equation (15) demonstrates that for a standard positive monotone creep kernel, the phase diagram will be decreasing monotonically. Hence, such kernels cannot describe the sharp variation of the phase velocity $C_p(\omega)$ near the resonance point shown in Fig. 1.

Analytical analysis shows that for the kernel $K(t)$ in the form $K(t) = A\exp(-\alpha t)$, the only possible way of satisfying the experimental data mentioned above is to consider the parameter α as a complex number. This in turn leads to the oscillating kernels of the type $\exp(-at)\cos bt$ or $\exp(-at)\sin bt$.

5.2. *Oscillating wave shapes*

Let us pass to another typical experimental phenomenon, the so-called oscillating wave shapes during a shock-wave propagation in laminated or fibre-reinforced composites (Fig. 2). These are the results of wave propagation through laminated steel–epoxy plate specimens presented in Lundergan and Drumheller (1971). A symmetric shock created a unidimensional deformation state in the specimen, which existed for about 9μs after the shock. The higher profile on Fig. 2 corresponds to the impulse load 5.27 kbar whereas the lower one corresponds to the impulse load 2.92 kbar. Vibrations presented in Fig. 2 are related with the effect of the resonance due to the periodical structure of the material.

It should be noted that dynamic tests on composites are more simple and accurate in realization than the harmonic ones. The reason for this is that ultrasonic experiments pose

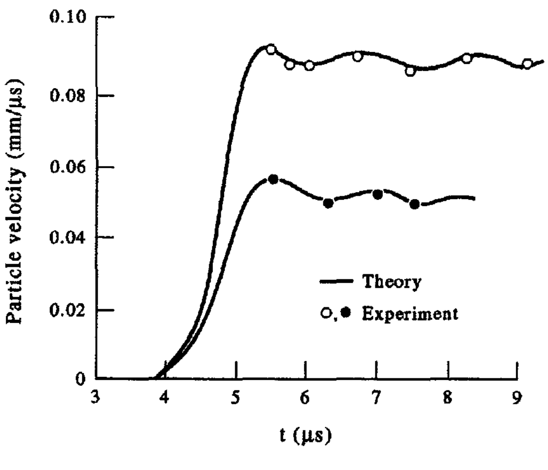

Fig. 2.

a lot of difficulties in the measuring, especially in resonance zones, which are the most interesting area to study. This obstacle can be easily avoided in shock wave tests.

Let us analyse the problem of a unidirectional wave propagation in a semi-infinite viscoelatic body with the constitutive law [eqn (16)] and with the boundary condition:

$$\sigma_{(x=0)} = \sigma_o H(t), \tag{17}$$

where $H(t)$ is the Heaviside step function. Then, after the Laplace transformation with respect to the variable $y = (t - x/c)$, we will obtain the following expression for the image of σ:

$$\bar{\sigma} = \frac{\sigma_o}{p} \exp\left(-\frac{px}{c}(\sqrt{1+\bar{K}} - 1) \right). \tag{18}$$

By realizing an asymptotic expansion for large values of the Laplace variable p [which in fact means for the small values of $(t\text{-}x/c)$], we have

$$\bar{\sigma} \cong \frac{\sigma_o}{p} \exp\left(-\frac{px\bar{K}}{2c} \right). \tag{19}$$

Now, by inverting the Laplace transform, one can see that traditional forms of the kernel K give only smooth monotone profiles and hence cannot describe the wave shape oscillations shown in Fig. 2. Indeed, for the creep kernel K, for example in the form $K(t) = A\exp(-\alpha t)$, one obtains from eqn (21) that

$$\bar{\sigma} \approx \frac{\sigma_o}{p} \exp\left(-\frac{x}{2c} \right) \exp\left(\frac{\alpha x}{2pc} \right). \tag{20}$$

Asymptotic inversion of eqn (20) gives, in terms of the Bessel functions,

$$\sigma \approx \sigma_o \exp\left(-\frac{x}{2c} \right) J_o(2\sqrt{(t - x/c)(-\alpha x/2c)}). \tag{21}$$

From here it follows that in order to describe, at least qualitatively, the experimental data presented in Fig. 2, factor α must be a complex number. For example, let us take the kernel of the form

$$K(t) = A\exp(-at)\sin bt. \tag{22}$$

Then eqn (21) will give

$$\sigma \approx \sigma_o J_o(2\sqrt{(t - x/c)xb/2c}). \tag{23}$$

By remembering Sections 5.1 and 5.2, we can conclude that the experimental phenomena given in Figs 1 and 2 can be qualitatively described in terms of homogeneous viscoelastic models only by means of generalized creep–relaxation functions. This was proposed first in Hazanov (1976).

5.3. *Data processing*

Let us now pass to the quantitative description of the presented experimental results. The data processing was realized with the creep kernel in the form of eqn (22). First, its thermodynamic admissibility must be verified.

It is easy to see that the thermodynamic condition (3) is equivalent to the positivity of the sin–Fourier transform of the creep kernel, $K_s > 0$. Eventually, the chosen kernel [eqn (22)] satisfies this condition. Indeed, direct computation gives

$$K_s(\omega) = \frac{2\,ab\,\omega}{((a-b)^2+\omega^2))((a+b)^2+\omega^2)}. \tag{24}$$

This expression is positive for all positive parameters a and b. The kernel eqn (22) is thermodynamically admissible.

For the geometric dispersion effect (Al-W fibre composite) the data processing gave the following values of the parameters of the kernel in the constitutive equation:

$$A = 1.835, \quad a = 0.5, \quad b = 4.17\cdot 10^6. \tag{25}$$

Parameter b, which is additional via the classic exponential kernel, has, in this case, a precise physical sense—it corresponds to the resonance frequency of the material. The parameter c/b in its turn corresponds to the dimension of the period of the composite microstructure (here 1 mm).

Figure 1 shows that the points predicted by the proposed model correspond to the experimental ones. The values of the kernel parameters for the dynamic tests (Fig. 2) have values of approximately of the same order. For instance, for the lower profile,

$$A = 8.486, \quad a = 2.127, \quad b = 4.69\cdot 10^6. \tag{26}$$

Comparison of the "experimental" and "theoretical" points again gives encouraging results.

Finally, one can conclude that a homogeneous viscoelastic constitutive law with even the simplest generalized creep kernel of the form of eqn (22), can describe rather accurately experimental data on wave propagation in fibre-reinforced and laminated composites.

6. POSSIBLE APPLICATIONS

A vast field of alternative applications of the generalized creep–relaxation kernels can be indicated. Let us begin with multi-transitional effects in wood [see Huet (1988)] where, in the phase Cole–Cole plane $(E_1(\omega)-E_2(\omega))$, experimental thermodynamic diagrams of the material usually have the form of a set of three consequent arcs. This shows a non-self-consistency of the traditional modelling of wood as of a viscoelastic material. As it is well known [see, for example, Christensen (1971)], such a thermodynamic phase diagram for a homogeneous viscoelastic material is a single arc, placed between the initial and the long term moduli. This is why it was proposed in Huet (1988) to describe this multi-transitional effect in wood by means of a chain of classic viscoelastic models with singular parabolic elements. This evidently poses numerous computational problems. However, it was shown in Hazanov and Zeiter (1993) that such a multi-transitional effect in wood can be easily explained qualitatively and quantitatively by means of only one viscoelastic Volterra model, but this time with generalized relaxation functions or kernels of heredity.

Among another possible applications of the proposed theory, one can recall the behaviour of numerous materials with oscillating and decreasing stress–strain diagrams (biologic materials, softening of concrete in fracture). Furthermore, we must mention here heterogeneous materials (composites), where the use of generalized creep–relaxation functions in homogenized constitutive equations gives a unique possibility to describe several important phenomena specific for composites. These are, for instance, negative Poisson ratios in cellular structures with inverted cells, described in Gibson and Ashby (1988), and the geometric dispersion and oscillating wave profiles studied above.

7. CONCLUSION

Thermodynamic study enables one to estimate the limits of the class of thermodynamically admissible creep–relaxation functions in linear viscoelasticity. Analysis of the Breuer–Onat work positivity condition is achieved and a new dissipation criterion [inequality (10)] is obtained.

It is shown that the class of thermodynamically admissible creep–relaxation functions is larger than the traditional set of monotone positive definite functions used in applications. In addition to these this class includes the generalized creep–relaxation functions, such as locally non-monotone and even non-positive ones. This involves the effect of the "negative local viscosity". It is demonstrated that for admissible generalized functions, the global viscosity of the body nevertheless always remains positive.

The classic fading memory principle is revised in order to treat the notion of the "fading memory" in a more general, integral sense. This is illustrated on the example of the creep kernel $K(t) = A \exp(-at) \sin(bt)$.

The generalized creep–relaxation functions proposed above can be obtained even in terms of classical mechanical spring–dashpot models, by inserting inertial terms into the latter, for example simple masses.

It is shown that the class of generalized creep–relaxation functions turns out to be quite effective in different domains of the mechanics of materials, especially in the homogenization of heterogeneous bodies and of composite materials. This concerns the dynamics of composites, the decreasing stress–strain diagrams, the effect of negative Poisson ratios in composites of cellular structure, the multi-transitional effect on thermomechanical behaviour of wood and the strain-softening in the fracture of concrete. One can mention here as well geologic structures where laying and bedding greatly influence the overall behaviour of the medium. Thus, an inhomogeneous material with relatively simple constitutive laws of the composants can be described quite efficiently by a homogeneous model of a more complicated physical nature (in our case elastoviscoplastic equations with generalized creep–relaxation functions).

The notion of the "negative viscosity" introduced above must be interpreted in a purely statistical sense, as is done in hydromechanics, where an analogous notion is applied to the modelling of turbulent streams by means of averaged "laminar" constitutive laws [see Starr (1968)].

The proposed concept of the "equivalence" between the geometry (inhomogeneity) and the physics (viscosity) can give numerous important applications, for instance by opening new avenues in the construction of universal models for solids.

Acknowledgements—We are grateful to C. Huet for fruitful discussions and to the National Swiss Foundation for Scientific Research for financial support for this work. Grant Nos 21-27962.89 and 20-32206.91.

REFERENCES

Barker, L. M. (1971). A model for stress wave propagation in composite materials. *J. Comp. Mater.* **4,** 140–164.

Breuer, S. and Onat, E. (1962). On uniqueness in linear viscoelasticity. *Q. Appl. Math.* **19**(4), 355–359.

Christensen, R. (1971). *Theory of Viscoelasticity. Introduction.* Academic Press, New York.

Gibson, L. J. and Ashby, M. F. (1988). *Cellular Solids.* Pergamon Press, Oxford.

Gurtin, M. E. and Herrera, I. (1965). On dissipation inequalities and linear viscoelasticity. *Q. Appl. Math.* **23,** 235–245.

Hazanov, S. (1976). On one dynamic model for composites. *Mekh. Polym.* **2,** 364–367 (in Russian).

Hazanov, S. and Zeiter, P. (1993). A new viscoelastic model for multitransition effect in wood. In *Wood: Plasticity and Damage* (Edited by C. Birkinshaw *et al.*), pp. 27–34. Proc. Int. Workshop on Wood Mechanics, Ireland.

Huet, C. (1988). Le fluage du bois en flexion: rôles de la température et de l'humidité. *Annales ITBTP* **469,** 35–53.

Lundergan, C. D. and Drumheller D. S. (1971). Propagation of stress waves in a laminate plate composite. *J. Appl. Physics* **42**(2), 669–675.

Rabotnov, Y. (1977). *Elements of Hereditary Mechanics of Solids.* Nauka, Moscow (in Russian).

Starr P. V. (1968). *Physics of Negative Viscosity Phenomena.* McGraw-Hill, New York.

Sutherland, H. J. and Lingle, R. (1972). Geometric dispersion of acoustic waves by a fibrous composite. *J. Compos. Mater.* **10,** 490–504.